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Cooperative Differential Games Strategies for Active Aircraft Protection from a Homing Missile

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Cooperative pursuit–evasion strategies are derived for a team composed of two agents. The specific problem of interest is that of protecting a target aircraft from a homing missile. The target aircraft performs evasive maneuvers and launches a defending missile to intercept the homing missile. The problem is analyzed using a linear quadratic differential game formulation for arbitrary-order linear players' dynamics in the continuous and discrete domains. Perfect information is assumed. The analytic continuous and numeric discrete solutions are presented for zero-lag adversaries' dynamics. The solution of the game provides 1) the optimal cooperative evasion strategy for the target aircraft, 2) the optimal cooperative pursuit strategy for the defending missile, and 3) the optimal strategy of the homing missile for pursuing the target aircraft and for evading the defender missile. The obtained guidance laws are dependent on the zero-effort miss distances of two pursuer–evader pairs: homing missile with target aircraft and defender missile with homing missile. Conditions for the existence of a saddle-point solution are derived and the navigation gains are analyzed for various limiting cases. Nonlinear two-dimensional simulation results are used to validate the theoretical analysis. The advantages of cooperation are shown. Compared with a conventional one-on-one guidance law, cooperation significantly reduces the maneuverability requirements from the defending missile.

I. Introduction

MODERN interceptor missiles present a significant threat to today's air, land, and navy vehicles, both civil and military. In the air arena, current countermeasure systems such as flares and chaff rely on decoying the incoming missile threat and may not provide adequate protection against modern, highly sophisticated, missiles. A promising alternative to current countermeasure technology could be an aircraft self-defense system that actively defeats an interceptor missile by launching a defending missile to intercept it. The agility demands from the antimissile missile may be very high in order to obtain superiority over the highly maneuverable attacking missile. A cooperative evasion and pursuit strategy may be incorporated in such a scenario in order to aid the aircraft and its defending missile in the cooperative engagement with the attacking missile. In this paper we focus on deriving such cooperative evasion and pursuit strategies.

Optimal control theory is usually applied for developing guidance laws for one-on-one pursuer–evader engagements, assuming perfect information [1]. The classical proportional navigation [2] (PN) guidance law is the optimal guidance law for intercepting a nonmaneuvering target by a pursuer with ideal dynamics; and the same is true for augmented proportional navigation [3] (APN), if the target performs a constant maneuver. An extension of APN for a pursuer with first-order dynamics is the well-known optimal guidance law [4] (OGL). Developing an evasive strategy for a target using optimal control theory requires an assumption on the future pursuer's behavior. Knowing the guidance law of the incoming interceptor, the aircraft can perform the optimal evasion strategy. Such an analysis of a two-dimensional interception scenario, where the interceptor is assumed to use PN guidance, was performed in [5]. A model with linear kinematics, bounded accelerations, and ideal pursuer dynamics was analyzed. It was found that the optimal evasion strategy has a bang–bang structure. In [6] the same model was investigated, but with nonlinear kinematics.

Target maneuvers are independently controlled; thus, future target maneuvers cannot be predicted. As a consequence, a deterministic optimal control formulation of such problems may not be appropriate. The mathematical framework for analyzing conflicts controlled by two independent agents is in the realm of dynamic games [7]. For such problems a zero-sum pursuit–evasion game can be formulated, requiring information only on the maneuver capability of the target and not its expected maneuver until engagement termination. The solution to the problem, if it exists, provides simultaneously the optimal pursuit and evasion maneuvers for the two adversaries. Such a linear quadratic formulation of a scenario between two players with ideal dynamics for both adversaries was investigated in [8]. The formulation includes a quadratic penalty on the terminal relative displacement (miss distance) and on the control effort of the adversaries, thus implicitly taking into account control saturation. Such a formulation is often called a linear quadratic differential game (LQDG). The optimal guidance law was found to have the same structure and can be viewed as a generalization of PN. The difference is the higher gain that is dependent on the maneuverability of the target relative to the missile. The solution was extended to include first-order strictly proper pursuer and target dynamics [9] and first-order biproper dynamics of the pursuer [10].

If perfect intercept cannot be achieved for a given scenario, a finite weight on the miss distance has to be selected in the formulation of the problem, such that a min-max solution with a finite miss distance exists. The obtained guaranteed miss distance against any feasible target maneuver is dependent on the adversaries' maneuverability and the initial conditions. Provided that the interceptor's warhead lethal radius is larger than this guaranteed cost, then interception can be guaranteed, at least under the assumption of perfect information.

In [11] it was proposed to use a defender missile to protect a target from a homing missile. The kinematic relations of three bodies (target, missile, and defender) moving in the plane around a collision course were studied in [12]. For the analysis it was assumed that the

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target is nonmaneuvering, no saturation limits exist for the missile and the defender, both the missile and the defender have ideal dynamics, and perfect information is available both to the missile about the target and to the defender about the missile.

If some of the agents have identical objectives, then they can be grouped in a team and actually serve as a single player having multiple controls. A two-team three-person game of antimissile ship defense was considered in [13]. The authors posed the problem in the framework of a two-person game by assuming that the defended ship is not maneuvering to avoid the incoming threat. Nonetheless, it imposes trajectory restrictions on the incoming interceptor, as it needs to avoid being intercepted by the defending missile while still eventually hitting the ship.

A two-team dynamic game, named the lady and the bodyguard versus the bandit, was posed in [14]. The bandit's objective is to capture the lady, while the objective of the lady and her bodyguard is to prevent it. The bodyguard is trying to intercept the bandit before his arrival to the proximity of the lady. An approach to the formulation and solution of the game using multi-objective optimization was presented. The game was specified as protecting a target from an attacking missile by a defender, and the required acceleration levels of the players were assessed [15]. The formulation and solution were given for linear continuous systems [14,15]. The formal solution was given in terms of a numerical solution of a Riccati equation with impulse function utilization in the indices in order to reflect the defender's disappearance after its interception by the attacking missile. In [16] the game was reformulated in the discrete domain, thus avoiding the use of an impulse function. A related problem between two pursuers and one evader was studied in [17] using a linear quadratic formulation.

Recently, the three-body target–missile–defender scenario received considerable attention. In [18] the kinematics of line-of-sight guidance with a maneuvering launch platform (defended aircraft) was investigated. Based on the kinematic results, a guidance law for the defended aircraft, cooperating with a command-to-line-of-sight guided defender, in order to maximize the attacker to defender lateral acceleration ratio, was proposed and studied analytically and via simulation. In [19] optimal-control-based cooperative evasion and pursuit strategies were derived for the aircraft and its defending missile for the case where the attacking missile uses a known linear guidance strategy. Limiting cases were analyzed in which the attacking missile uses PN, APN, or OGL. The optimal one-on-one target evasion strategy from such guidance laws was derived as well. It was shown that depending on the initial conditions, and in contrast to the optimal one-on-one evasion strategy, the optimal cooperative target maneuver is either constant or arbitrary. In [20] the case where the target and its defender share noisy measurements on the attacking missile was investigated. A nonlinear adaptation of a multiple-model adaptive estimator was used in order to identify the guidance law and the guidance parameters of the incoming homing missile. A matched defender's missile guidance law was optimized to the identified homing-missile guidance law. It uses cooperation between the aerial target and the defender missile. The cooperation stems from the fact that the defender knows the future evasive maneuvers to be performed by the protected target and thus can anticipate the maneuvers it will induce on the incoming homing missile. Moreover, the target performs a maneuver that minimizes the control effort requirements from the defender. The estimator and guidance law were combined in a multiple-model adaptive control configuration, showing excellent homing performance.

In this paper we propose perfect-information cooperative LQDG (CLQDG) guidance laws for the three agents engaged in the target–missile–defender scenario. The scenario is posed as a two-team game and, as such, we do not assume knowing, or need to identify, the missile's guidance strategy. Arbitrary-order linear dynamics are assumed for the adversaries, and for simplifying the derivation, the order of the problem is reduced using the *terminal projection* transformation. Analytic continuous and numeric discrete solutions are derived for zero-lag adversaries' dynamics. Conditions for the existence of a saddle-point solution are derived and the navigation gains are analyzed for various limiting cases. A nonlinear two-dimensional simulation is used to validate the theoretical analysis. The remainder of this paper is organized as follows: In the next section the mathematical models of the engagement are provided. Then the cooperative strategies for the aircraft and defending missile are derived, as well as the optimal strategy for the attacking missile. Analytical analysis of the guidance laws is provided in the following section. Then a simulation analysis is presented, followed by concluding remarks. In the Appendix we present an example of the mathematical treatment of the defender's disappearance after the end of its interception stage with the attacking missile.

II. Problem Formulation

The problem consists of three entities: an evading target T, an attacking missile M, and a defending missile D. The attacking missile is chasing the evading target that at some point launches a defending missile to intercept the incoming threat. We deal with the endgame of such a scenario, initiated when the defending missile is launched toward the attacking missile. Obviously, the engagement between the defending missile and the attacking missile (denoted as MD) is planned to terminate before that between the attacking missile and the evading target (denoted as MT). In this section we first present the nonlinear kinematics equations of the interception problem, which will be used in the nonlinear simulation in Sec. V. Then the linearized equations used for the guidance laws derivation are presented. The engagement is analyzed in two dimensions.

A. Nonlinear Kinematics

In Fig. 1 a schematic view of the planar endgame geometry is shown, where $X_I - O_I - Z_I$ is a Cartesian inertial reference frame. The notations MT and MD denote the attacking missile with evading target and attacking missile with defending missile duos, respectively. The speed, normal acceleration, and flight-path angles are denoted by V, a, and γ, respectively. The range between the adversaries is r; and λ is the angle between a line of sight (LOS) and the X_I axis.

Neglecting the gravitational force, the engagement kinematics between the attacking missile and the evading target, expressed in a polar coordinate system (r, λ) attached to the target, are

$$\dot{r}_{MT} = -V_M \cos(\gamma_M + \lambda_{MT}) - V_T \cos(\gamma_T - \lambda_{MT}) \tag{1a}$$

$$\dot{\lambda}_{MT} = [V_M \sin(\gamma_M + \lambda_{MT}) - V_T \sin(\gamma_T - \lambda_{MT})]/r_{MT} \tag{1b}$$

Similarly, the engagement kinematics equations between the defending missile and the attacking one are

$$\dot{r}_{MD} = -V_M \cos(\gamma_M + \lambda_{MD}) - V_D \cos(\gamma_D - \lambda_{MD}) \tag{2a}$$

$$\dot{\lambda}_{MD} = [V_M \sin(\gamma_M + \lambda_{MD}) - V_D \sin(\gamma_D - \lambda_{MD})]/r_{MD} \tag{2b}$$

During the endgame, the adversaries are assumed to move at constant speeds.

The path angles of the adversaries evolve based on

$$\dot{\gamma}_i = a_i/V_i; \qquad i = \{M, T, D\} \tag{3}$$

We assume that during the endgame the dynamics of each agent can be represented by arbitrary-order linear equations:

$$\dot{\boldsymbol{x}}_i = \mathbf{A}_i \boldsymbol{x}_i + \mathbf{B}_i u_i'; \qquad i = \{M, T, D\} \tag{4}$$

$$a_i = \mathbf{C}_i \boldsymbol{x}_i + d_i u_i'; \qquad i = \{M, T, D\} \tag{5}$$

where $\boldsymbol{x}_i$ is the state vector of an agent's internal state variables with $\dim(\boldsymbol{x}_i) = n_i$ and u_i' is its controller.

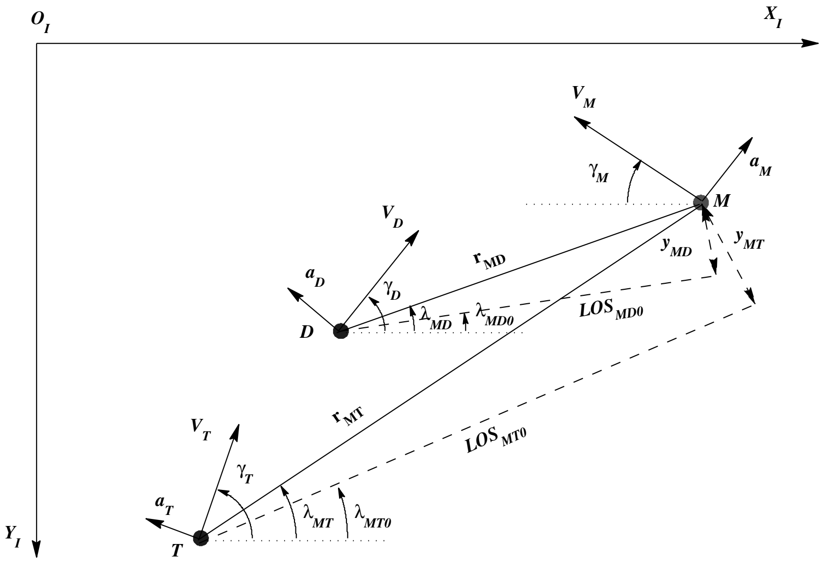

Fig. 1 Planar engagement geometry.

B. Linearized Kinematics

In this engagement with three entities, we have two collision triangles: one in the scenario between the attacking missile and evading target and the other in the scenario between the defending missile and the attacking missile. We deal with the endgame and assume that flight near these nominal collision triangles can be assumed. In the simulation section we continuously perform linearization (i.e., every time cycle) around the current collision triangles.

We denote y_{MT} as the relative displacement between M and T, normal to LOS_{MT_0}; similarly, y_{MD} is the relative displacement between M and D, normal to LOS_{MD_0}.

Let us denote the missile's and target's accelerations normal to LOS_{MT} as a_{MN} and a_{TN}, respectively; similarly, the defender's acceleration normal to LOS_{MD} is denoted as a_{DN}, satisfying

$$\begin{cases} a_{MN} = a_M \cos(\gamma_{M0} + \lambda_{MT_0}) = \mathbf{C}_M \boldsymbol{x}_M \cos(\gamma_{M_0} + \lambda_{MT_0}) + d_M u_M \\ a_{TN} = a_T \cos(\gamma_{T0} - \lambda_{MT_0}) = \mathbf{C}_T \boldsymbol{x}_T \cos(\gamma_{T_0} - \lambda_{MT_0}) + d_T u_T \\ a_{DN} = a_D \cos(\gamma_{D0} - \lambda_{MD_0}) = \mathbf{C}_D \boldsymbol{x}_D \cos(\gamma_{D_0} - \lambda_{MD_0}) + d_D u_D \end{cases} \tag{6}$$

where u_M, u_T, and u_D are the agents' controllers normal to the corresponding LOS:

$$\begin{cases} u_M = u'_M \cos(\gamma_{M_0} + \lambda_{MT_0}) \\ u_T = u'_T \cos(\gamma_{T_0} - \lambda_{MT_0}) \\ u_D = u'_D \cos(\gamma_{D_0} - \lambda_{MD_0}) \end{cases} \tag{7}$$

Remark 1. We assume that the adversaries' velocity vectors are not perpendicular to the corresponding LOS; i.e., the problem remains controllable during the entire game duration.

The state vector of the linearized problem is

$$\boldsymbol{x} = [\boldsymbol{x}_{MT}^T \quad \boldsymbol{x}_{MD}^T \quad \boldsymbol{x}_M^T]^T \tag{8}$$

where

$$\boldsymbol{x}_{MT} = [y_{MT} \quad \dot{y}_{MT} \quad \boldsymbol{x}_T^T]^T \tag{9}$$

$$\boldsymbol{x}_{MD} = [y_{MD} \quad \dot{y}_{MD} \quad \boldsymbol{x}_D^T]^T \tag{10}$$

and $\dim(\boldsymbol{x}) = 4 + n_M + n_T + n_D$

The equations of motion are

$$\dot{\boldsymbol{x}} = \begin{cases} \dot{\boldsymbol{x}}_{MT} = \begin{cases} \dot{\boldsymbol{x}}_{MT_1} = \boldsymbol{x}_{MT_2} \\ \dot{\boldsymbol{x}}_{MT_2} = a_{TN} - a_{MN} \\ \dot{\boldsymbol{x}}_T = \mathbf{A}_T \boldsymbol{x}_T + \mathbf{B}_T u_T / \cos(\gamma_{T_0} - \lambda_{MT_0}) \end{cases} \\ \dot{\boldsymbol{x}}_{MD} = \begin{cases} \dot{\boldsymbol{x}}_{MD_1} = \boldsymbol{x}_{MD_2} \\ \dot{\boldsymbol{x}}_{MD_2} = a_{MN} \cos(\lambda_{MT_0} - \lambda_{MD_0}) - a_{DN} \\ \dot{\boldsymbol{x}}_D = \mathbf{A}_D \boldsymbol{x}_D + \mathbf{B}_D u_D / \cos(\gamma_{D_0} - \lambda_{MD_0}) \end{cases} \\ \dot{\boldsymbol{x}}_M = \mathbf{A}_M \boldsymbol{x}_M + \mathbf{B}_M u_M / \cos(\gamma_{M_0} + \lambda_{MT_0}) \end{cases} \tag{11}$$

The entire set of equations can now be written as

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}[u_T \quad u_D]^T + \mathbf{C} u_M \tag{12}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{MT} & [0]_{(n_T+2)\times(n_D+2)} & \mathbf{C}_{MT} \\ [0]_{(n_D+2)\times(n_T+2)} & \mathbf{A}_{MD} & \mathbf{C}_{MD} \\ [0]_{n_M\times(n_T+2)} & [0]_{n_M\times(n_D+2)} & \mathbf{A}_M \end{bmatrix} \tag{13}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{MT} & [0]_{(n_T+2)\times 1} \\ [0]_{(n_D+2)\times 1} & \mathbf{B}_{MD} \\ [0]_{n_M\times 1} & [0]_{n_M\times 1} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} [0 \quad d_M \quad [0]_{1\times n_T}]^T \\ [0 \quad d_M \cos(\lambda_{MT_0} - \lambda_{MD_0}) \quad [0]_{1\times n_T}]^T \\ \mathbf{B}_M / \cos(\gamma_{M_0} + \lambda_{MT_0}) \end{bmatrix} \tag{14}$$

and

$$\mathbf{A}_{MT} = \begin{bmatrix} 0 & 1 & [0]_{1\times n_T} \\ 0 & 0 & \mathbf{C}_T \cos(\gamma_{T_0} - \lambda_{MT_0}) \\ [0]_{n_T\times 1} & [0]_{n_T\times 1} & \mathbf{A}_T \end{bmatrix}$$

$$\mathbf{B}_{MT} = \begin{bmatrix} 0 \\ d_T \\ \mathbf{B}_T / \cos(\gamma_{T_0} - \lambda_{MT_0}) \end{bmatrix} \tag{15}$$

$$\mathbf{A}_{MD} = \begin{bmatrix} 0 & 1 & [0]_{1\times n_D} \\ 0 & 0 & -\mathbf{C}_D \cos(\gamma_{D_0} - \lambda_{MD_0}) \\ [0]_{n_D\times 1} & [0]_{n_D\times 1} & \mathbf{A}_D \end{bmatrix}$$

$$\mathbf{B}_{MD} = \begin{bmatrix} 0 \\ -d_D \\ \mathbf{B}_D / \cos(\gamma_{D_0} - \lambda_{MD_0}) \end{bmatrix} \tag{16}$$

$$\mathbf{C}_{MT} = \begin{bmatrix} 0 \\ -\mathbf{C}_M \cos(\gamma_{M_0} + \lambda_{MT_0}) \\ [0]_{n_T\times 1} \end{bmatrix}$$

$$\mathbf{C}_{MD} = \begin{bmatrix} 0 \\ \mathbf{C}_M \cos(\gamma_{M_0} + \lambda_{MD_0}) \cos(\lambda_{MT_0} - \lambda_{MD_0}) \\ [0]_{n_D\times 1} \end{bmatrix} \tag{17}$$

with [0] denoting a matrix of zeros with appropriate dimensions.

C. Timeline

The initial range between the attacking missile and the evading target is r_{MT_0}. Similarly, the initial range between the attacking missile and the defending missile is r_{MD_0}. Under the linearization assumption of small deviations from a collision triangle, the interception time is fixed, satisfying

$$t_{fMT} = -r_{MT_0}/\dot{r}_{MT_0} = r_{MT_0}/[V_M \cos(\gamma_{M_0} + \lambda_{MT_0}) + V_T \cos(\gamma_{T_0} - \lambda_{MT_0})] \tag{18}$$

Similarly,

$$t_{fMD} = -r_{MD_0}/\dot{r}_{MD_0} = r_{MD_0}/[V_M \cos(\gamma_{M_0} + \lambda_{MD_0}) + V_D \cos(\gamma_{D_0} - \lambda_{MD_0})] \tag{19}$$

We define Δt as the time difference between interceptions,

$$\Delta t = t_{fMT} - t_{fMD} \tag{20}$$

and we require that the MD engagement terminates before that of MT. Thus, $\Delta t > 0$ ($t_{fMD} < t_{fMT}$). Also, the defender missile ceases to exist after $t = t_{fMD}$. Thus, we enforce $u_D = 0 \; \forall \; t \geq t_{fMD}$ (see the Appendix for an example).

D. Order Reduction

To enable an easier formulation of the problem and also to reduce the problem's order we will use the following transformation [21], denoted by some as terminal projection:

$$\begin{cases} Z_{MT}(t) = \mathbf{D}_{MT}\mathbf{\Phi}(t_{fMT}, t)\boldsymbol{x}(t) \\ Z_{MD}(t) = \mathbf{D}_{MD}\mathbf{\Phi}(t_{fMD}, t)\boldsymbol{x}(t) \end{cases} \tag{21}$$

where $\mathbf{\Phi}$ is the transition matrix associated with Eq. (12), and $\mathbf{D}_{MT}$, $\mathbf{D}_{MD}$ are constant vectors,

$$\begin{cases} \mathbf{D}_{MT} = [1 \quad [0]_{1\times(n_T+n_D+n_M+3)}] \\ \mathbf{D}_{MD} = [[0]_{1\times(n_T+2)} \quad 1 \quad [0]_{1\times(n_D+n_M+1)}] \end{cases} \tag{22}$$

and $Z_{MT}(t)$ and $Z_{MD}(t)$ define the zero-effort miss (ZEM) vector:

$$\mathbf{Z}(t) = [Z_{MT}(t) \quad Z_{MD}(t)]^T \tag{23}$$

The derivative with respect to time of $Z_{MT}(t)$ is

$$\begin{aligned} \dot{Z}_{MT}(t) &= \mathbf{D}_{MT}[\dot{\mathbf{\Phi}}(t_{fMT}, t)\boldsymbol{x} + \mathbf{\Phi}(t_{fMT}, t)\dot{\boldsymbol{x}}] \\ &= \mathbf{D}_{MT}\mathbf{\Phi}(t_{fMT}, t)(\mathbf{B}[u_T \quad u_D]^T + \mathbf{C}u_M) \\ &= \mathbf{D}_{MT}\mathbf{\Phi}(t_{fMT}, t)([\mathbf{B}_{MT}^T \quad [0]_{1\times(n_D+n_M+2)}]^T u_T + \mathbf{C}u_M) \end{aligned} \tag{24}$$

Similarly,

$$\dot{Z}_{MD}(t) = \mathbf{D}_{MD}\mathbf{\Phi}(t_{fMD}, t)([[0]_{1\times(n_T+2)} \quad \mathbf{B}_{MD}^T \quad [0]_{1\times n_M}]^T u_D + \mathbf{C}u_M) \tag{25}$$

Note that both derivatives, $\dot{Z}_{MD}(t)$ and $\dot{Z}_{MT}(t)$, are state-independent.

Remark 2. In addition to reducing the order of the problem, the two variables of the new state vector $\mathbf{Z}(t)$ have an important physical meaning. $Z_{MT}(t)$ is the zero-effort miss in the attacking-missile/evading-target engagement, which is the miss distance if neither the attacking missile nor the evading target will apply any control from the current time onward. Similarly, $Z_{MD}(t)$ is defined in the defending/attacking-missile engagement.

In our study, in order to reflect the defending missile's disappearance after termination of the engagement with the attacking missile, we force $Z_{MD}(t)$ to remain constant for $t_{fMD} < t \leq t_{fMT}$ by special treatment of the dynamic equations (see the Appendix for an example):

$$Z_{MD}(t) \equiv_{t\geq t_{fMD}} Z_{MD}(t_{fMD}) \tag{26}$$

E. Discrete Equations of Motion

We will also define and solve the problem in the discrete domain, in order to obtain the reference for validating the new analytic solution of the cooperative guidance law.

The discrete version of Eqs. (24) and (25) is

$$\mathbf{Z}_d(i+1) = \mathbf{Z}_d(i) + \Gamma_d(i)\mathbf{u}(i) \tag{27}$$

where $\mathbf{Z}_d(i)$ and $\mathbf{u}(i)$ are the discrete state and the discrete control vectors,

$$\mathbf{Z}_d(i) = [Z_{MT}(i) \quad Z_{MD}(i)]^T \tag{28}$$

$$\mathbf{u}(i) = [u_M(i) \quad u_T(i) \quad u_D(i)]^T \tag{29}$$

and the $\Gamma_{\mathbf{d}}(\mathbf{i})$ matrix is defined as

$$\Gamma_d(i) = T_s \begin{bmatrix} \mathbf{D}_{MT}\mathbf{\Phi}(t_{fMT}, iT_s)[\mathbf{C}|\mathbf{B}] \\ \mathbf{D}_{MD}\mathbf{\Phi}(t_{fMD}, iT_s)[\mathbf{C}|\mathbf{B}] \end{bmatrix} \tag{30}$$

where T_s is the sampling time.

Remark 3. The zero-effort miss vector of Eq. (23) is obtained from the homogeneous solution of the differential equation (12). Thus, its state transition matrix is an identity matrix; consequently, with zero control ($\mathbf{u}(i) = 0$) we obtain in Eq. (27) that $\mathbf{Z}_d(i+1) = \mathbf{Z}_d(i)$.

F. Cost Function

1. *Continuous Time*

The quadratic cost function chosen for this problem is

$$J = -\frac{\alpha_{MT}}{2}y_{MT}^2(t_{fMT}) + \frac{\alpha_{MD}}{2}y_{MD}^2(t_{fMD}) + \frac{\beta_T}{2}\int_0^{t_{fMT}} u_T^2 \,\mathrm{d}t + \frac{\beta_D}{2}\int_0^{t_{fMD}} u_D^2 \,\mathrm{d}t - \frac{1}{2}\int_0^{t_{fMT}} u_M^2 \,\mathrm{d}t \tag{31}$$

where the weights α_{MT}, α_{MD}, β_D, and β_D are nonnegative.

The problem involving the three agents is posed as a linear quadratic differential game between two teams. One team (the minimizers) is composed of the evading target and its defender missile, and the attacking missile belongs to the other team (the maximizer). Thus, the controls u_T and u_D are issued so as to minimize this cost function, and u_M is used to maximize it:

$$\min_{[u_T u_D]} \max_{u_M} J \tag{32}$$

Furthermore, each participant wishes, simultaneously, to minimize its energy expenditure formulated as an integral cost. Since for

$t_{fMD} < t \le t_{fMT}$ we actually have a standard one-on-one engagement, we only need to find the cooperative strategies between the target and the defender up to t_{fMD}.

Using the ZEM variables, the cost function from Eq. (31) can be expressed as

$$J_Z = -\frac{\alpha_{MT}}{2} Z_{MT}^2(t_{fMT}) + \frac{\alpha_{MD}}{2} Z_{MD}^2(t_{fMD}) + \frac{\beta_T}{2}\int_0^{t_{fMT}} u_T^2\,\mathrm{d}t + \frac{\beta_D}{2}\int_0^{t_{fMD}} u_D^2\,\mathrm{d}t - \frac{1}{2}\int_0^{t_{fMT}} u_M^2\,\mathrm{d}t \tag{33}$$

As for $t \ge t_{fMD}$, $Z_{MD}(t) = Z_{MD}(t_{fMD})$, and $u_D(t) = 0$, the problem to be solved may be simplified to being dependent on a single final time:

$$J_Z = -\frac{\alpha_{MT}}{2} Z_{MT}^2(t_{fMT}) + \frac{\alpha_{MD}}{2} Z_{MD}^2(t_{fMT}) + \frac{1}{2}\int_0^{t_{fMT}} (\beta_T u_T^2 + \beta_D u_D^2 - u_M^2)\,\mathrm{d}t \tag{34}$$

2. Discrete Time

The cost function from Eq. (33) can be rewritten in the discrete domain as

$$J_d = -\frac{\alpha_{MT}}{2} Z_{MT}^2(N_{fMT}) + \frac{\alpha_{MD}}{2} Z_{MD}^2(N_{fMD}) + \frac{T_s}{2}\left(\sum_{i=0}^{N_{fMT}-1} (-u_M^2(i) + \beta_T u_T^2(i)) + \sum_{i=0}^{N_{fMD}-1} \beta_D u_D^2(i)\right) \tag{35}$$

where

$$N_{fj} = t_{fj}/T_s, \qquad j = \{MT, MD\} \tag{36}$$

In the discrete domain we can easily insert the terminal costs into the running cost with varying coefficients:

$$J_d = \frac{1}{2}\sum_{i=0}^{N_{fMT}} \{\mathbf{Z}_d^T(i)\mathbf{Q}(i)\mathbf{Z}_d(i) + \mathbf{u}^T(i)\mathbf{R}(i)\mathbf{u}(i)\} \tag{37}$$

where

$$\mathbf{Q}(i) = \begin{cases} \begin{bmatrix} -\alpha_{MT} & 0 \\ 0 & 0 \end{bmatrix}, & i = N_{fMT} \\ \begin{bmatrix} 0 & 0 \\ 0 & \alpha_{MD} \end{bmatrix}, & i = N_{fMD} \\ [0]_{2\times 2}, & \text{else} \end{cases} \tag{38}$$

$$\mathbf{R}(i) = \begin{cases} T_s \begin{bmatrix} -1 & 0 & 0 \\ 0 & \beta_T & 0 \\ 0 & 0 & \beta_D \end{bmatrix}, & i < N_{fMT} \\ [0]_{3\times 3}, & i = N_{fMT} \end{cases} \tag{39}$$

III. Guidance Laws

The defined problem could be divided into two phases: before the termination of the engagement between the defending and attacking missiles ($t < t_{fMD}$), when we need to find the optimal cooperative strategy for the evading target and the defender and the optimal strategy for the attacking missile, and from that time onward ($t_{fMD} \le t < t_{fMT}$), when we actually have a standard one-on-one engagement. It is assumed that the success of the defending missile is not guaranteed. Therefore, the game will terminate at t_{fMT}.

A. CLQDG Analytical Continuous Solution

1. Optimal Controllers

The Hamiltonian of the problem is

$$H = \lambda_1 \dot{Z}_{MT}(t) + \lambda_2 \dot{Z}_{MD}(t) + \tfrac{1}{2}(\beta_T u_T^2 + \beta_D u_D^2 - u_M^2) \tag{40}$$

The adjoint equations are

$$\begin{cases} \dot{\lambda}_1(t) = -\frac{\partial H}{\partial Z_{MT}} = 0; \\ \dot{\lambda}_2(t) = -\frac{\partial H}{\partial Z_{MD}} = 0; \end{cases} \quad \begin{cases} \lambda_1(t_{fMT}) = \frac{\partial J_Z}{\partial Z_{MT}(t_{fMT})} = -\alpha_{MT} Z_{MT}(t_{fMT}) \\ \lambda_2(t_{fMT}) = \frac{\partial J_Z}{\partial Z_{MD}(t_{fMT})} = \alpha_{MD} Z_{MD}(t_{fMT}) \end{cases} \tag{41}$$

yielding the solution:

$$\begin{cases} \lambda_1(t) = -\alpha_{MT} Z_{MT}(t_{fMT}) \\ \lambda_2(t) = \alpha_{MD} Z_{DM}(t_{fMD}) \end{cases} \tag{42}$$

The optimal controllers for the defender, target, and the missile satisfy

$$\begin{cases} (u_D^*, u_T^*) = \arg_{(u_D, u_T)} \min(H) \\ u_M^* = \arg_{u_M} \max(H) \end{cases} \tag{43}$$

The Hamiltonian is differentiable with respect to the controllers for any $t \le t_{fMT}$. Therefore, by equating the derivatives of the Hamiltonian with respect to the players' controllers to zero, we obtain the open-loop optimal controllers:

$$\begin{cases} u_M^* = -\left(\alpha_{MT} Z_{MT}(t_{fMT}) \frac{\partial \dot{Z}_{MT}(t)}{\partial u_M} - \alpha_{MD} Z_{DM}(t_{fMD}) \frac{\partial \dot{Z}_{MD}(t)}{\partial u_M}\right) \\ u_T^* = \frac{1}{\beta_T}\left(\alpha_{MT} Z_{MT}(t_{fMT}) \frac{\partial \dot{Z}_{MT}(t)}{\partial u_T} - \alpha_{MD} Z_{DM}(t_{fMD}) \frac{\partial \dot{Z}_{MD}(t)}{\partial u_T}\right) \\ u_D^* = \frac{1}{\beta_D}\left(\alpha_{MT} Z_{MT}(t_{fMT}) \frac{\partial \dot{Z}_{MT}(t)}{\partial u_D} - \alpha_{MD} Z_{DM}(t_{fMD}) \frac{\partial \dot{Z}_{MD}(t)}{\partial u_D}\right) \end{cases} \tag{44}$$

Then, substituting the open-loop optimal controllers from Eq. (44) into $\dot{Z}_{MT}(t)$ and $\dot{Z}_{MD}(t)$ from Eqs. (24) and (25) and integrating from t to t_{fMT} and to t_{fMD}, respectively, we obtain two coupled algebraic equations for $Z_{MT}(t_{fMT})$ and for $Z_{MD}(t_{fMD})$. Next, solving these two equations and substituting the result back into Eq. (44), we obtain the closed-form solution:

$$\begin{cases} u_M^*(t) = \frac{N'_M(t) Z_{MT}(t)}{t_{\mathrm{go}MT}^2} + \frac{\Gamma'_M(t) Z_{MD}(t)}{t_{\mathrm{go}MD}^2} \\ u_T^*(t) = \frac{N'_T(t) Z_{MT}(t)}{t_{\mathrm{go}MT}^2} + \frac{\Gamma'_T(t) Z_{MD}(t)}{t_{\mathrm{go}MD}^2} \\ u_D^*(t) = \frac{\Gamma'_D(t) Z_{MT}(t)}{t_{\mathrm{go}MT}^2} + \frac{N'_D(t) Z_{MD}(t)}{t_{\mathrm{go}MD}^2} \end{cases} \tag{45}$$

where $t_{\mathrm{go}MT}$ is the time-to-go in the engagement between the attacking missile and the evading target, and $t_{\mathrm{go}MD}$ is the time-to-go in the engagement between the attacking and defending missiles. These variables are defined as nonnegative (see the Appendix):

$$t_{\mathrm{go}i} = \begin{cases} t_{fi} - t, & t \le t_{fi} \\ 0, & t > t_{fi} \end{cases}, \qquad i = \{MT, MD\} \tag{46}$$

N'_i and Γ'_i are the effective navigation gains ($i = \{M, T, D\}$). Note that each navigation gain has a different effect:

1) N'_M is responsible for pursuing the target: i.e., decreasing $Z_{MT}(t)$ for $t < t_{fMT}$.

2) Γ'_M is responsible for avoiding the defending missile: i.e., increasing $Z_{MD}(t)$ for $t < t_{fMD}$.

3) N'_T is responsible for avoiding the attacking missile: i.e., increasing $Z_{MT}(t)$ for $t < t_{fMT}$.

4) Γ'_T is responsible for assisting the defending missile to intercept the attacking missile by acting like a bait: i.e., decreasing $Z_{MD}(t)$ for $t < t_{fMD}$.

5) N'_D is responsible for pursuing the attacking missile: i.e., decreasing $Z_{MD}(t)$ for $t < t_{fMD}$.

6) Γ_D' is responsible for assisting the target to avoid the attacking missile by causing the attacking missile to avoid the defender: i.e., increasing $Z_{MT}(t)$ for $t < t_{fMD}$.

2. Ideal Dynamics Case

To obtain an analytic solution for the strategies, we will assume that the adversaries have ideal dynamics: i.e., zero lag. The zero-effort miss variables for this case are defined as follows:

$$\begin{cases} Z_{MT}(t) = y_{MT}(t) + \dot{y}_{MT}(t)(t_{fMT} - t) \\ Z_{MD}(t) = y_{MD}(t) + \dot{y}_{MD}(t)(t_{fMD} - t) \end{cases} \tag{47}$$

and the time derivatives are

$$\begin{cases} \dot{Z}_{MT}(t) = -t_{\text{go}MT} \cdot u_M + t_{\text{go}MT} \cdot u_T \\ \dot{Z}_{MD}(t) = t_{\text{go}MD} \cdot u_M - t_{\text{go}MD} \cdot u_D \end{cases} \tag{48}$$

Using Eq. (48) in Eq. (44) yields the closed-loop solution of Eq. (45) with the following navigation gains:

$$\begin{cases} N_M' = (\alpha_{MT}\phi_{z11}t_{\text{go}MT} + \alpha_{MD}\phi_{z21}t_{\text{go}MD}\cos(\lambda_{MT_0} - \lambda_{MD_0}))t_{\text{go}MT}^2 \\ \Gamma_M' = (\alpha_{MT}\phi_{z12}t_{\text{go}MT} + \alpha_{MD}\phi_{z22}t_{\text{go}MD}\cos(\lambda_{MT_0} - \lambda_{MD_0}))t_{\text{go}MD}^2 \\ N_T' = (\alpha_{MT}/\beta_T)\phi_{z11}t_{\text{go}MT}^3 \\ \Gamma_T' = (\alpha_{MT}/\beta_T)\phi_{z12}t_{\text{go}MT}t_{\text{go}MD}^2 \\ N_D' = (\alpha_{MD}/\beta_D)\phi_{z22}t_{\text{go}MD}^3 \\ \Gamma_D' = (\alpha_{MD}/\beta_D)\phi_{z21}t_{\text{go}MD}t_{\text{go}MT}^2 \end{cases} \tag{49}$$

where

$$\begin{cases} \phi_{z11} = \left(1 + \frac{\alpha_{MD}}{3}(1/\beta_D - \cos(\lambda_{MT_0} - \lambda_{MD_0}))t_{\text{go}MD}^3\right)\Big/\Delta_z \\ \phi_{z12} = -\alpha_{MD}\left(\frac{t_{\text{go}MD}^3}{3} + \frac{t_{\text{go}MD}^2}{2}\Delta t\right)\cos(\lambda_{MT_0} - \lambda_{MD_0})\Big/\Delta_z \\ \phi_{z21} = \alpha_{MT}\left(\frac{t_{\text{go}MD}^3}{3} + \frac{t_{\text{go}MD}^2}{2}\Delta t\right)\Big/\Delta_z \\ \phi_{z22} = \left(1 + \frac{\alpha_{MT}}{3}(1 - 1/\beta_T)t_{\text{go}MT}^3\right)\Big/\Delta_z \end{cases} \tag{50}$$

and

$$\begin{aligned} \Delta_z = {} & \left(1 + \frac{\alpha_{MT}}{3}(1 - 1/\beta_T)t_{\text{go}MT}^3\right) \\ & \times \left(1 + \frac{\alpha_{MD}}{3}(1/\beta_D - \cos(\lambda_{MT_0} - \lambda_{MD_0}))t_{\text{go}MD}^3\right) \\ & + \alpha_{MT}\alpha_{MD}\left(\frac{t_{\text{go}MD}^3}{3} + \frac{t_{\text{go}MD}^2}{2}\Delta t\right)^2\cos(\lambda_{MT_0} - \lambda_{MD_0}) \end{aligned} \tag{51}$$

B. CLQDG Discrete Numerical Solution

The Hamiltonian of the discrete problem is

$$\begin{aligned} H(i) = {} & \tfrac{1}{2}(\mathbf{Z}_d^T(i)\mathbf{Q}(i)\mathbf{Z}_d(i) + \mathbf{u}^T(i)\mathbf{R}(i)\mathbf{u}(i)) \\ & + \lambda^T(i+1)\mathbf{Z}_d(i+1) \end{aligned} \tag{52}$$

The adjoint equation is

$$\lambda^T(i) = -\frac{\partial H(i)}{\partial \mathbf{Z}_d(i)} \tag{53}$$

$$\begin{aligned} \lambda^T(i) &= \lambda^T(i+1) + \mathbf{Z}_d^T(i)\mathbf{Q}(i) \\ \lambda^T(N_{fMT}) &= \mathbf{Z}_d^T(N_{fMT})\mathbf{Q}(N_{fMT}) \end{aligned} \tag{54}$$

A stationary value of $H(i)$ with respect to $\mathbf{u}(i)$ will occur when we have

$$\frac{\partial H(i)}{\partial \mathbf{u}(i)} = \mathbf{u}(i)\mathbf{R}(i) + \lambda^T(i+1)\Gamma_d(i) = 0 \tag{55}$$

Consequently,

$$\mathbf{u}^*(i) = -\mathbf{R}^{-1}(i)\Gamma_d(i)\lambda^T(i+1) \tag{56}$$

and we obtain

$$\mathbf{Z}_d(i+1) = \mathbf{Z}_d(i) - \Gamma_d(i)\mathbf{R}^{-1}(i)\Gamma_d^T(i)\lambda(i+1) \tag{57}$$

Let us define the $\mathbf{P}(i)$ matrix, such that

$$\lambda(i) = \mathbf{P}(i)\mathbf{Z}_d(i) \tag{58}$$

Therefore, we obtain two equations:

$$\begin{cases} \mathbf{Z}_d(i+1) = [\mathbf{I} + \Gamma_d(i)\mathbf{R}^{-1}(i)\Gamma_d^T(i)\mathbf{P}(i+1)]^{-1}\mathbf{Z}_d(i) \\ \lambda(i) = \mathbf{P}(i+1)\mathbf{Z}_d(i+1) + \mathbf{Q}(i)\mathbf{Z}_d(i) \end{cases} \tag{59}$$

where $\mathbf{I}$ is an identity matrix with appropriate dimensions. Finally,

$$\begin{cases} \mathbf{u}^*(i) = -\mathbf{R}^{-1}(i)\Gamma_d(i)\mathbf{P}(i+1)[\mathbf{I} + \Gamma_d(i)\mathbf{R}^{-1}(i)\Gamma_d^T(i)\mathbf{P}(i)]^{-1}\mathbf{Z}_d(i), & i < N_{fMT} \\ \mathbf{P}(i) = \mathbf{P}(i+1)[\mathbf{I} + \Gamma_d(i)\mathbf{R}^{-1}(i)\Gamma_d^T(i)\mathbf{P}(i+1)]^{-1} + \mathbf{Q}(i), & i < N_{fMT} \\ \mathbf{P}(N_{fMT}) = \mathbf{Q}(N_{fMT}) \end{cases} \tag{60}$$

We obtain the optimal controllers by iterating $\mathbf{P}(i)$ backward from N_{fMT} to the current instant.

Let us define the matrix that will contain the coefficient of $\mathbf{Z}_d(i)$ from Eq. (60):

$$\mathbf{K}(i) = -\mathbf{R}^{-1}(i)\Gamma_d(i)\mathbf{P}(i+1)[\mathbf{I} + \Gamma_d(i)\mathbf{R}^{-1}(i)\Gamma_d^T(i)\mathbf{P}(i)]^{-1} \tag{61}$$

Therefore, the optimal controllers may be presented as follows:

$$\mathbf{u}^*(i) = \mathbf{K}(i)\mathbf{Z}_d(i) \tag{62}$$

Specifically, these controllers can be written as dependent on the corresponding ZEM and the navigation gains:

$$\begin{cases} u_M^*(i) = \frac{N'_M(i)Z_{MT}(i)}{(N_{fMT}-i)^2T_s^2} + \frac{\Gamma'_M(i)Z_{MD}(i)}{(N_{fMD}-i)^2T_s^2} \\ u_T^*(i) = \frac{N'_T(i)Z_{MT}(i)}{(N_{fMT}-i)^2T_s^2} + \frac{\Gamma'_T(i)Z_{MD}(i)}{(N_{fMD}-i)^2T_s^2} \\ u_D^*(i) = \frac{\Gamma'_D(i)Z_{MT}(i)}{(N_{fMT}-i)^2T_s^2} + \frac{N'_D(i)Z_{MD}(i)}{(N_{fMD}-i)^2T_s^2} \end{cases} \tag{63}$$

where the navigation gains are

$$\begin{cases} N'_M(i) = K_{(1,1)}(i)(N_{fMT}-i)^2T_s^2 \\ N'_T(i) = K_{(2,1)}(i)(N_{fMT}-i)^2T_s^2 \\ N'_D(i) = K_{(3,2)}(i)(N_{fMD}-i)^2T_s^2 \\ \Gamma'_M(i) = K_{(1,2)}(i)(N_{fMD}-i)^2T_s^2 \\ \Gamma'_T(i) = K_{(2,2)}(i)(N_{fMD}-i)^2T_s^2 \\ \Gamma'_D(i) = K_{(3,1)}(i)(N_{fMT}-i)^2T_s^2 \end{cases} \tag{64}$$

In the next section we will compare between the continuous and discrete solutions.

IV. Analytical Analysis

We will now analyze the CLQDG navigation gains of Eq. (49), derived assuming ideal dynamics. First, we will examine some limiting cases. Next, we will present a comparison between the results of the discrete and continuous derivations. Then the sufficient conditions for existence of the game solution, i.e., nonexistence of a conjugate point, will be presented.

A. Guidance Gains: Limiting Cases

1. Inactive Defender with Decisive Target and Missile

By choosing $\alpha_{MD} \to 0$ we no longer impose interception of the attacking missile and the defending one (N'_D, $\Gamma'_D \to 0$). Thus, the attacking missile does not need to avoid the defender ($\Gamma'_M \to 0$) and the target does not need to assist the defender ($\Gamma'_T \to 0$), resulting in

$$\begin{cases} N'_{M(\alpha_{MD}\to 0)} = \frac{3\alpha_{MT}\beta_T t_{goMT}^3}{(3\beta_T - \alpha_{MT}(1-\beta_T)t_{goMT}^3)}; & \Gamma'_{M(\alpha_{MD}\to 0)} = 0 \\ N'_{T(\alpha_{MD}\to 0)} = \frac{3\alpha_{MT} t_{goMT}^3}{(3\beta_T - \alpha_{MT}(1-\beta_T)t_{goMT}^3)}; & \Gamma'_{T(\alpha_{MD}\to 0)} = 0 \\ N'_{D(\alpha_{MD}\to 0)} = 0; & \Gamma'_{D(\alpha_{MD}\to 0)} = 0 \end{cases} \tag{65}$$

Therefore, the CLQDG guidance law [Eq (45)] degenerates to the classical one-on-one LQDG guidance law between the target and the attacking missile. There is no conjugate point for $\beta_T > 1$ (see [9]).

If we also choose $\alpha_{MT} \to \infty$, which dictates zero miss for the attacking missile, the navigation gains further degenerate to

$$N'_{M(\alpha_{MD}\to 0, \alpha_{MT}\to\infty)} = \frac{3\beta_T}{(\beta_T - 1)}$$
$$N'_{T(\alpha_{MD}\to 0, \alpha_{MT}\to\infty)} = \frac{3}{(\beta_T - 1)} \tag{66}$$

By considering a nonmaneuvering target scenario ($\beta_T \to \infty$) for the missile, we obtain the well-known optimal PN guidance law with $N'_M = 3$.

2. Target Confident in its Defender with Nondecisive Attacking Missile

Similar to the previous section, by choosing $\alpha_{MT} \to 0$ we no longer impose the interception of the target by the attacking missile ($N'_M, N'_T \to 0$), the target does not try to assist its defender ($\Gamma'_T \to 0$), and the defender is not dependent on the target's existence ($\Gamma'_D \to 0$). Thus, CLQDG degenerates to a one-on-one LQDG between the defending missile (pursuer) and the attacking missile (evader), resulting in

$$\begin{cases} N'_{M(\alpha_{MT}\to 0)} = 0; & \Gamma'_{M(\alpha_{MT}\to 0)} = \frac{3\alpha_{MD}\beta_D t_{goMD}^3}{(3\beta_D - \alpha_{MD}(\beta_D-1)t_{goMD}^3)} \\ N'_{T(\alpha_{MT}\to 0)} = 0; & \Gamma'_{T(\alpha_{MT}\to 0)} = 0 \\ N'_{D(\alpha_{MT}\to 0)} = \frac{3\alpha_{MD} t_{goMD}^3}{(3\beta_D - \alpha_{MD}(\beta_D-1)t_{goMD}^3)}; & \Gamma'_{D(\alpha_{MT}\to 0)} = 0 \end{cases} \tag{67}$$

There is no conjugate point for $\beta_D < 1$ (see [9]).

3. Two-Team Game: Perfect Target Interception

For a perfect interception of the evading target by the attacking missile, with some account for the interception of the attacking missile by the defending one, we would require $\alpha_{MT} \to \infty$ and a finite α_{MD}. Figure 2 presents the navigation gains versus physical time to interception of the attacking missile ($t_{goMD\,\text{true}}$) for different values of the weight α_{MD}.

As the weight α_{MD} increases, i.e., there is a greater demand for interception of the attacking missile by the defending one, the

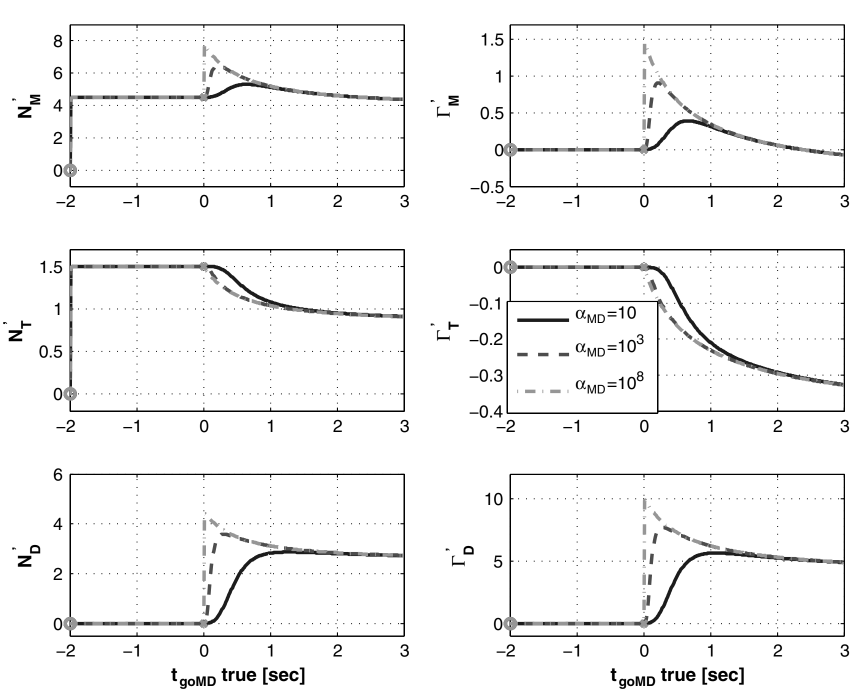

Fig. 2 Effective navigation gains for a perfect missile–target interception ($\alpha_{MT} \to \infty$), $\beta_T = 3$, $\beta_D = 1/3$, $t_{fMD} = 3$ s, and $t_{fMT} = 5$ s.

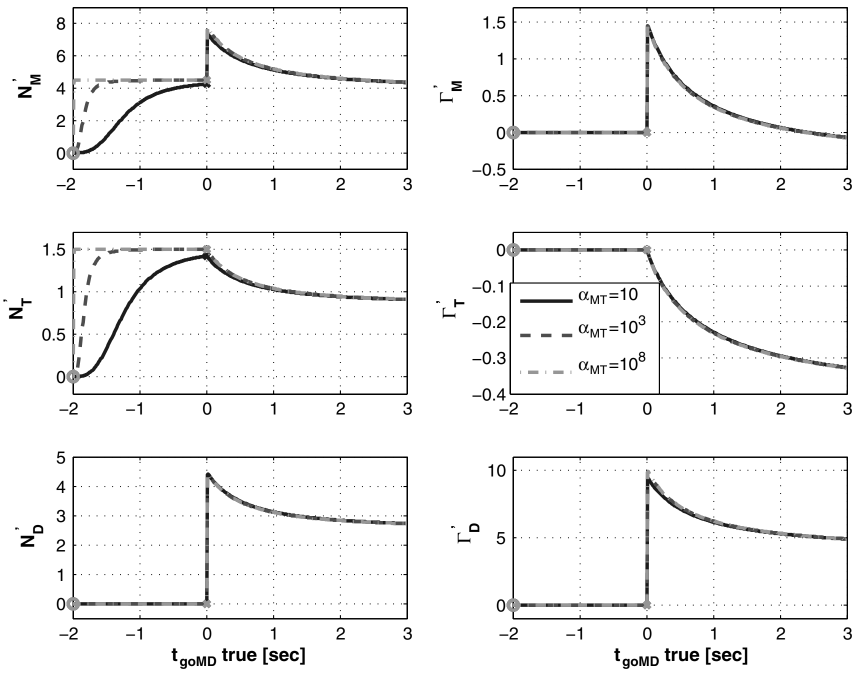

Fig. 3 Effective navigation gains for a perfect defending- and attacking-missile interception ($\alpha_{MD} \to \infty$), $\beta_T = 3$, $\beta_D = 1/3$, $t_{fMD} = 3$ s, and $t_{fMT} = 5$ s.

navigation gains grow in absolute manner toward the interception instance. The reason is that the defender tries harder to intercept the attacking missile (larger N'_D) and at the same time assists the target to avoid the attacking missile (larger Γ'_D); and the attacking missile makes a greater effort to avoid the defender (larger Γ'_M), while trying to intercept the target (larger N'_M); in contrast, N'_T and Γ'_T are hardly affected by α_{MD}.

4. Two-Team Game: Perfect Attacker Interception

For perfect interception of the attacking missile by the defending one, with some account for interception of the target by the attacking missile, we would require $\alpha_{MD} \to \infty$ and a finite α_{MT}. Figure 3 shows that the weight α_{MT} has a small influence on N'_M, N'_T and Γ'_D and no effect at all on the other navigation gains before interception of the attacking missile by the defending one. We can see from both Figs. 2 and 3 that after interception of the attacking missile by the defending one ($t_{goMD\,\text{true}} < 0$) the values of the navigation gains correspond to the missile–target one-on-one solution, since we assume that the defender ceases to exist.

B. Discrete and Continuous Solution Comparison

Equations (49) and (64) provide the values of the navigation gains in the continuous and discrete domains, respectively. Next, we present a comparison between the two solutions. As the basis for the comparison we chose the scenario with the parameters $\alpha_{MT} \to \infty$, $\alpha_{MD} \to \infty$, $\beta_T = 3$, $\beta_D = 1/3$, which was already presented in Figs. 2 and 3.

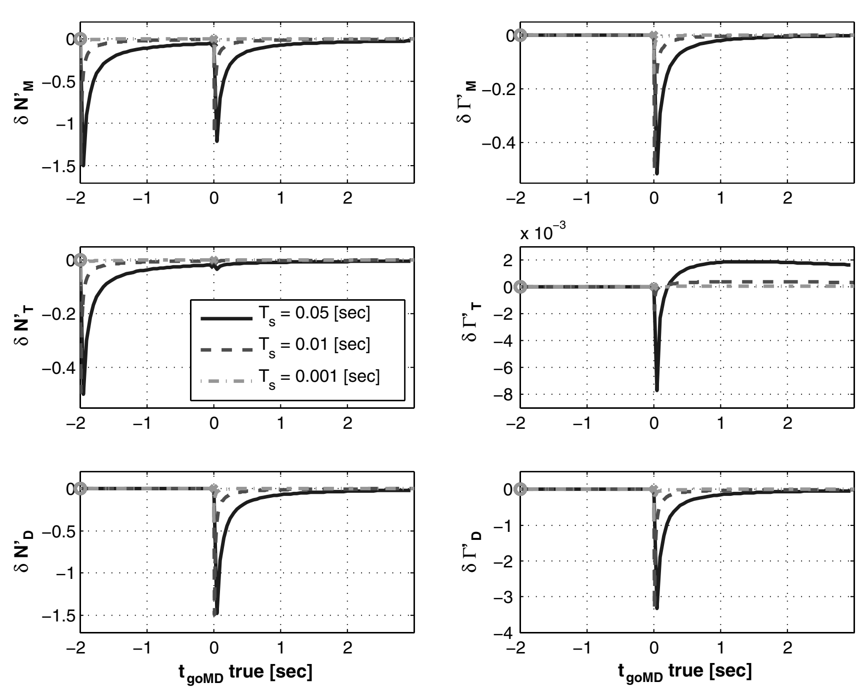

Fig. 4 Comparison of the effective navigation gains from the discrete numeric and analytic continuous solutions, $\alpha_{MT} \to \infty$, $\alpha_{MD} \to \infty$, $\beta_T = 3$, $\beta_D = 1/3$, $t_{fMD} = 3$ s, and $t_{fMT} = 5$ s.

Figure 4 presents the difference between the effective navigation gains obtained from the analytic continuous solution and the ones obtained from the numeric discrete solution with different sampling times T_s (0.05, 0.01, and 0.001 s).

It is apparent that the numeric discrete solution converges to the analytic continuous one, as the sampling time is reduced, thus validating the results.

C. Sufficient Conditions for Existence of a Game Solution

The nonexistence of a conjugate point has been shown to be a sufficient condition for the existence of a saddle-point solution to the game. In such a case (of a saddle-point solution) if one of the adversaries deviates from its optimal strategy, then it cannot gain. A conjugate point does not exist if and only if the navigation gains provided in Eqs. (49–51) are finite for all $t_0 < t \leq t_{fMT}$ [9]. It is easy to see that the navigation gains are unbounded when $\Delta_z \to 0$, where Δ_z is defined in Eq. (51).

As was explained earlier, the game has two phases, before and after the end of the engagement between the attacking missile and the defending missile at t_{fMD}, because the defender ceases to exist for $t_{fMD} \leq t \leq t_{fMT}$. Therefore, the general solution of the optimal controllers [Eq. (45)] degenerates into the one-on-one game solution, which was presented in Eq. (65). Consequently, Δ_z also changes according to these two phases:

$$\begin{cases} \Delta_z = \Delta_z(t_{\mathrm{go}MD}, \Delta t, \alpha_{MT}, \alpha_{MD}, \beta_T, \beta_D, \lambda_{MT_0}, \lambda_{MD_0}), & t_0 < t < t_{fMD} \\ \Delta_z = \Delta_z(t_{\mathrm{go}MT}, \alpha_{MT}, \beta_T), & t_{fMD} \leq t < t_{fMT} \end{cases} \tag{68}$$

For the second phase ($t_{fMD} \leq t < t_{fMT}$), it was shown earlier that the sufficient condition for the existence of a saddle-point solution is $\beta_T > 1$. Further, it was found that this condition is valid for the entire game timeline $t_0 < t \leq t_{fMT}$.

The weights β_T and β_D are design parameters associated with the expected relative maneuvering capability of the target and the defending missile. Similarly, α_{MT} and α_{MD} are weights associated with homing performance requirements from the attacking missile and the defending missile, respectively. All these weights were defined as nonnegative, according to the cost function definition in Eq. (31).

Summarizing, we obtain the following sufficient conditions for the existence of a saddle-point solution:

$$\begin{cases} \alpha_{MT} \geq 0, & \alpha_{MD} \geq 0 \\ \beta_T^{\mathrm{cr}} > \beta_T > 1, & \beta_D^{\mathrm{cr}} > \beta_D > 0 \end{cases} \tag{69}$$

where β_D^{cr} and β_T^{cr} are the maximum values of β_D and β_T that do not cause a conjugate point for the entire game duration; i.e., Δ_z does not have roots for $t_{\mathrm{go}MD} > 0$.

The general analysis of Δ_z of the first phase ($t_0 < t < t_{fMD}$) is too complicated, because it depends on many variables. Thus, in order to present the cooperative phase conjugate-point analysis in a relatively compact form, we will make some simplifying assumptions. First, we will assume that the angle between the two lines of sight (LOS_{MD} and LOS_{MT}) is relatively small, and therefore $\cos(\lambda_{MT_0} - \lambda_{MD_0}) \to 1$ in Eq. (51). This assumption is quite reasonable, because as we will see from the nonlinear simulation results, the defender, fired from the target, tends to fly alongside LOS_{MT} with relatively small deviations.

Investigating the CLQDG optimal controllers [Eq. (45)], it was found that in some cases β_D may be equal to, or even greater than, 1 without causing a conjugate point, in contrast to the classic LQDG [Eq. (67)]. Figure 5 presents the conditions on β_D^{cr} and β_T^{cr} for $\Delta t = 0.2$ s and for various values of $\alpha = \alpha_{MT} = \alpha_{MD}$. Similarly, Fig. 6 presents the conditions on β_D^{cr} and β_T^{cr} for $\alpha_{MT} = \alpha_{MD} = 10^5$ and for various values of Δt. According to these graphs, if we choose β_D and β_T below the critical line for particular α_{MT}, α_{MD}, and Δt, it will guarantee the problem solution without a conjugate point for the entire game duration.

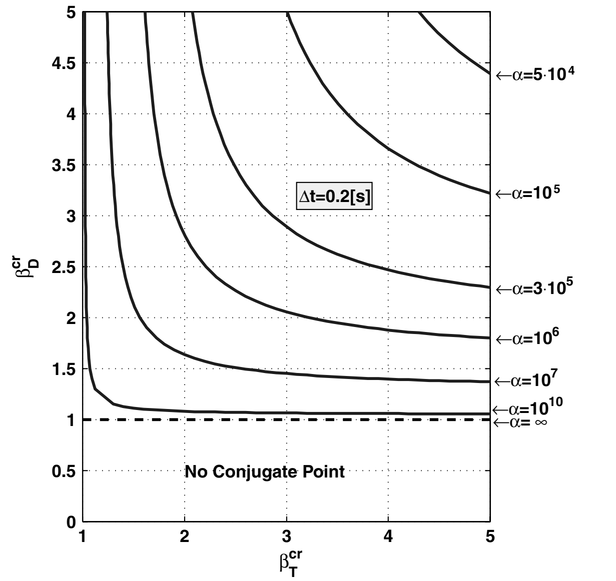

Fig. 5 Conditions on β_D^{cr} vs β_T^{cr} for CLQDG saddle-point solution for $\Delta t = 0.2$ s, and various values of $\alpha = \alpha_{MT} = \alpha_{MD}$.

One can note from the figures that when perfect interception is required ($\alpha_{MT} \to \infty$ and $\alpha_{MD} \to \infty$) or when the cooperative effect is very weak due to great difference between the final times ($\Delta t \to \infty$), the defender must not have a larger weight on its control relative to the attacking missile ($\beta_D \leq 1$). But if we are prepared to compromise the homing performance, i.e., not to require too-large values of α_{MT} and α_{MD}, we may use a more limited defender ($\beta_D > 1$) (see Fig. 5). Similarly, in the particular scenario, if the final times are tight, i.e., Δt is relatively small, it provides a strong cooperative effect and, consequently, a relatively limited defender might be used (see Fig. 6).

We may also conclude from the presented behavior of β_D^{cr} that as the target maneuverability increases (β_T^{cr} is reduced toward 1^+), the defender maneuverability requirements are reduced. For example, if we look at the line that corresponds to $\Delta t = 0.3$ s in Fig. 6 for the case of $\beta_T = 5$, the value of β_D must not exceed 2.2, and if the target is more maneuverable ($\beta_T = 2$), the defender may be much weaker ($\beta_D^{\mathrm{cr}} = 5$).

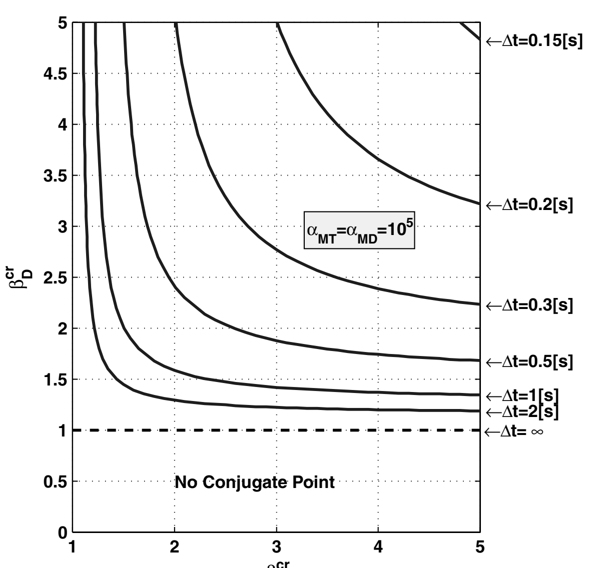

Fig. 6 Conditions on β_D^{cr} and β_T^{cr} for CLQDG saddle-point solution for $\alpha_{MT} = \alpha_{MD} = 10^5$ and various values of Δt.

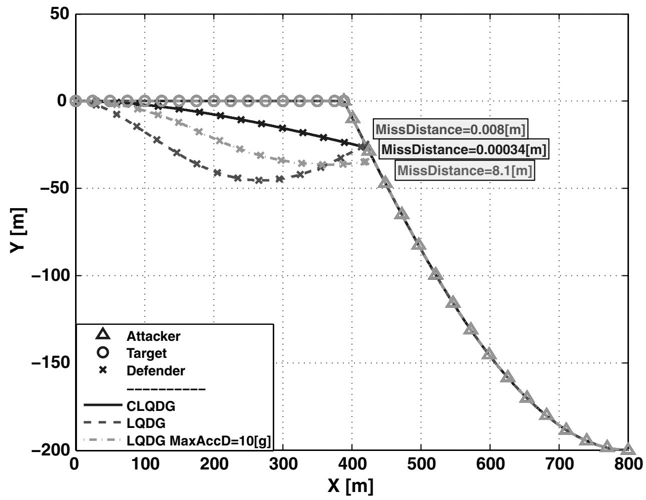

Fig. 7 Adversaries' trajectories when target does not maneuver, CLQDG vs LQDG, $\alpha_{MT} = \alpha_{MD} = 10^5$, $\beta_T \to \infty$, and $\beta_D = 1/3$.

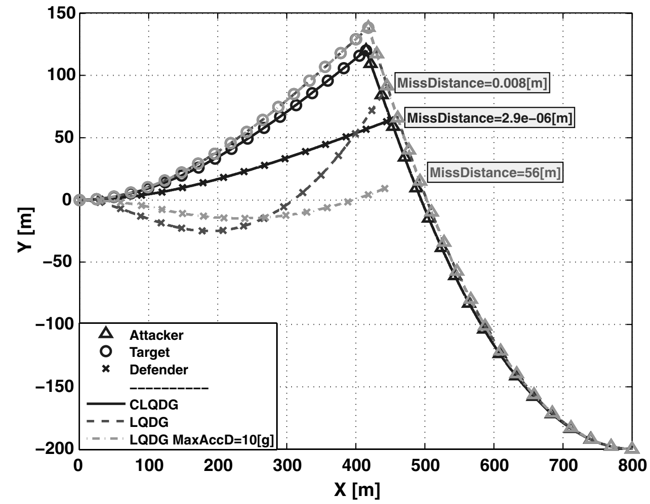

Fig. 9 Adversaries' trajectories, CLQDG vs LQDG, $\alpha_{MT} = \alpha_{MD} = 10^5$, $\beta_T = 3$, and $\beta_D = 1/3$.

V. Simulation Analysis

In this section the performance of the cooperative guidance law is investigated and compared with the classical one-on-one LQDG guidance law via simulation of the nonlinear kinematics. The engagement is initiated in a head-on scenario with some lateral shift, which represents the initial heading error for the attacking and defending missiles. For the sake of simplicity, the altitude of all the adversaries is assumed to remain constant throughout the engagement (i.e., a planar scenario). The initial displacement between the target and the attacking missile is $\Delta X = 800$ m and $\Delta Y = 200$ m. It is assumed that the target launches its defender at the beginning of the scenario. The speed of the target is $v_T = 250$ m/s and that of both missiles is $v_M = v_D = 300$ m/s. All the players have zero-lag dynamics. Thus, the miss distance is usually negligible for unsaturated sample runs and the comparison is with respect to the maneuver capabilities required from the adversaries.

Next, we will show three sample runs that demonstrate the following cooperation effects: 1) information sharing, 2) target cooperation, and 3) defenders' maneuverability limitations.

In the following figures a performance comparison of the CLQDG guidance law versus the classical LQDG guidance law with/without saturation is provided. For the sake of convenience, each line on the graphs is marked with a corresponding marker for each player [△: attacking missile, ○: evading target (aircraft), and ×: defending missile] in an isochronous way: i.e., each 0.1 s from the simulation beginning.

The navigation gains for the CLQDG guidance law are computed according to Eq. (49) for the three adversaries. For the classical LQDG guidance law the gains N'_M and N'_T are computed according to Eq. (65) for the attacking missile and the evading target, respectively, and the N'_D weight is computed according to Eq. (67) for the defender. The saturation was applied on the defender's normal-to-velocity acceleration.

A. Effect of Information Sharing

Figure 7 presents a sample run of the trajectories of the three adversaries, and Fig. 8 presents their normal-to-velocity acceleration profiles. In this scenario the evading target makes no maneuvers ($\beta_T \to \infty$) and completely relies on its defender (like a cargo or a passenger aircraft might).

The difference in the trajectory of the defender missile when implementing the CLQDG (solid lines) compared with the classical LQDG (dashed lines) is evident. The defender that uses the cooperative guidance law demonstrated superior performance, because it makes use of the information that the attacking missile intends to intercept the evading target and knows that the target will not maneuver. This helps the defender to head to the predicted interception point without making unnecessary maneuvers. Consequently, its maximum acceleration magnitude is less than 5 g, compared with around 50 g for the noncooperative case. Moreover, when the defender's acceleration is saturated with 10 g limit

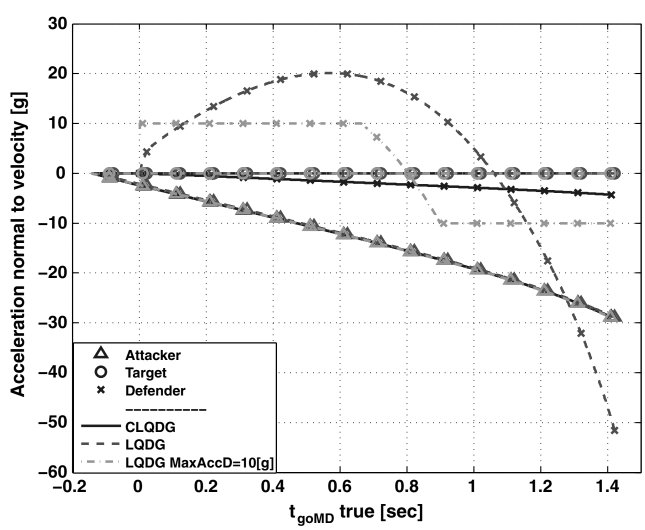

Fig. 8 Adversaries' accelerations when target does not maneuver, CLQDG vs LQDG, $\alpha_{MT} = \alpha_{MD} = 10^5$, $\beta_T \to \infty$, and $\beta_D = 1/3$.

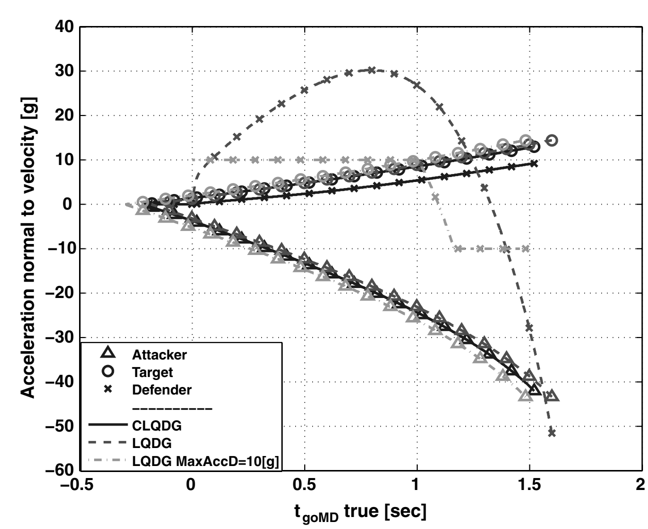

Fig. 10 Adversaries' accelerations, CLQDG vs LQDG, $\alpha_{MT} = \alpha_{MD} = 10^5$, $\beta_T = 3$, and $\beta_D = 1/3$.

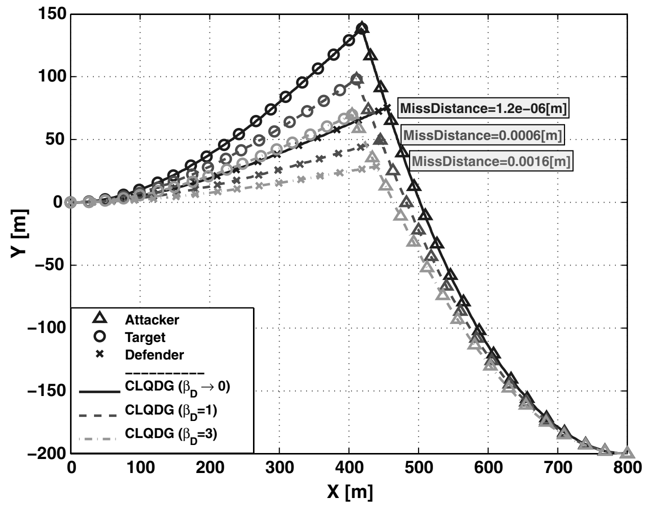

Fig. 11 Adversaries' trajectories for different defender's capabilities, CLQDG, $\alpha_{MT} = \alpha_{MD} = 10^5$, and $\beta_T = 3$.

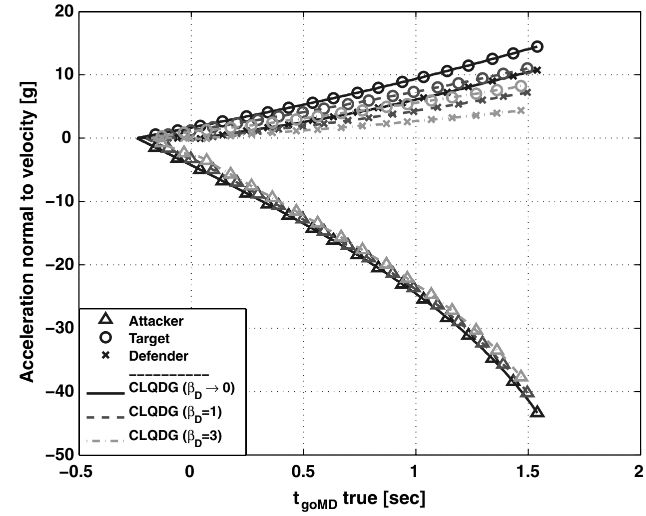

Fig. 12 Adversaries' accelerations for different defender's capabilities, CLQDG, $\alpha_{MT} = \alpha_{MD} = 10^5$, and $\beta_T = 3$.

(dashed–dotted lines), the miss distance, when using LQDG guidance law, is no longer negligibly small (8.1 m). The maneuvers of the attacking missile are almost identical for LQDG and CLQDG guidance laws.

B. Effect of Target Cooperation

Figures 9 and 10 present a similar scenario, but here the target makes evading maneuvers ($\beta_T = 3$). In addition to information sharing, as we saw in the previous scenario, the cooperative defender also gains from the target cooperation. The target evades slightly less from the attacking missile, in comparison with the noncooperative target, making the interception of the attacking missile easier for its defender. Because of this, the cooperative defender intercepts the threat at a distance of about 90 m from the target, maneuvering less than 10 g throughout all the flight, compared with the noncooperative unsaturated defender, which intercepts the attacker 70 m from the target, making extreme maneuvers up to about 52 g. When the saturation of 10 g is applied, the defender that uses LQDG guidance law fails to protect its target, passing the threat at a distance of 56 m.

C. Effect of Defender's Maneuverability

Figures 11–13 present the adversaries' trajectories, accelerations, and effective navigation gains, respectively, for three runs with various values of β_D $(0^+, 1, 3)$, which represents the influence of the defender's maneuverability on the CLQDG guidance law. In this scenario, according to Fig. 6, for $\Delta t \approx 0.2$ s and $\alpha_{MT} = \alpha_{MD} = 10^5$, we may use β_D up to a value of 5 without causing a conjugate point.

From Figs. 11 and 12, it is evident that the target's evading strategy depends on the defenders limitations; i.e., as the defender is more limited, the target evades less, assisting its defender to intercept the agile attacking missile, in order to succeed in the cooperative mission.

Now let us concentrate on the values of the effective navigation gains, presented in Fig. 13. In the first scenario (solid lines), due to the defender's large maneuver capability ($\beta_D \to 0$), the target does not need to assist it ($\Gamma'_T(\beta_D \to 0) \to 0$) and concentrates on evading the missile. Note that the defender's effective navigation gain $N'_D(\beta_D \to 0) = 3$ is constant throughout the entire flight, as in PN guidance. Moreover, as the defender is highly maneuverable, it is of no use for the attacking missile to evade the defender ($\Gamma'_M(\beta_D \to 0) \to 0$),

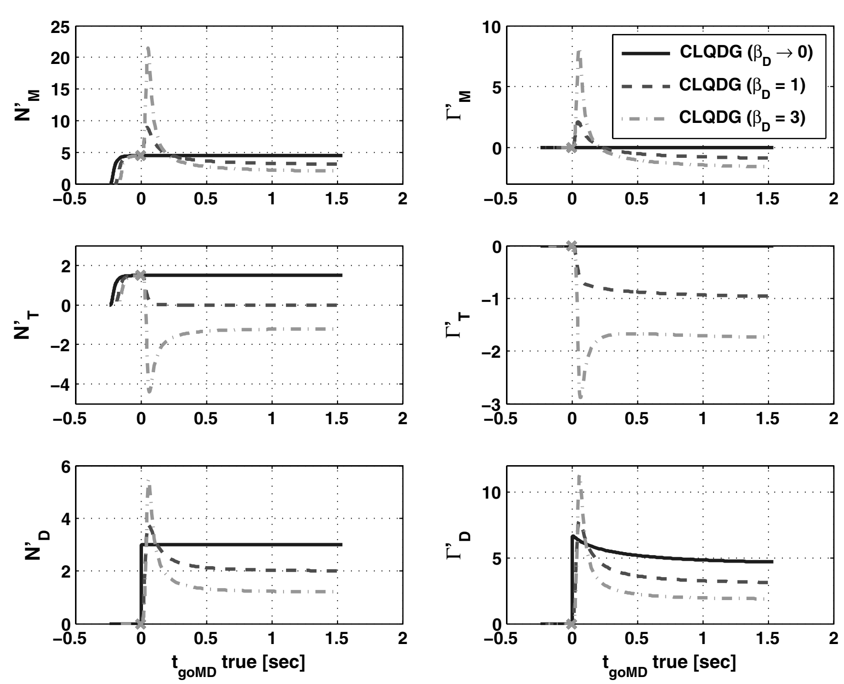

Fig. 13 Effective navigation gains for different defender's capabilities, CLQDG, $\alpha_{MT} = \alpha_{MD} = 10^5$, and $\beta_T = 3$.

knowing that it is much more capable and therefore the attacking missile only pursues the target.

In the second scenario (dashed lines), the defender's maneuver capability is reduced ($\beta_D = 1$). The target, being aware of its defender's limitations, behaves in a less aggressive way, in order to assist the defender. Note that the classic LQDG guidance law provides an optimal solution only when a pursuer (missile) is more capable than the evading target ($\beta_T > 1$).

In the third scenario (dashed–dotted lines) the defender is even more limited ($\beta_D = 3$). Nevertheless, CLQDG still has no conjugate point, unlike the classical LQDG. The target cooperates with its defender by acting more like a bait and performs even less aggressive evasive maneuvers. Note that during almost the entire defender's flight $N'_T(\beta_D = 3) < 0$, and just before the interception of the attacking missile by the defender, the sign switches to a positive one. This shows the target's strategy change from initially acting like a bait to performing an evasive strategy. As a result of such cooperation, the defender succeeds in intercepting the attacking missile with a small miss distance, in spite of being much less maneuverable than the attacking missile.

VI. Conclusions

An optimal cooperative guidance law has been derived for an interception engagement in which a defending missile is fired from an evading aircraft to defend it from an incoming homing missile. The problem has been analyzed using a linear quadratic differential game formulation for arbitrary-order linear players' dynamics in the continuous and discrete domains. The closed-form analytic solution has been obtained for zero-lag adversaries' dynamics and was validated by a comparison with the discrete numeric solution. The navigation gains were studied and their behavior was analyzed for various limiting cases. Theoretic conditions for the existence of a saddle-point solution in the differential game were also derived.

Nonlinear simulation results showed the benefits of the cooperative over noncooperative linear quadratic differential games guidance law. Because of the cooperation, the defender heads directly to the predicted interception point, without making unnecessary maneuvers. This dramatically reduces the control effort requirements from the defending missile. The cooperative target, in addition to performing evasive maneuvers, also assists the defender by acting like a bait, making the attacking missile easy to intercept, even when the defender's maneuverability is relatively limited. It was shown that when the final times are tight, i.e., the difference between the interception times of attacking missile with target aircraft and defender missile with attacking missile is relatively small, the cooperative effect is strong and, consequently, a relatively limited defender might be used.

The capability demonstrated by the cooperative strategies to protect an evading aircraft from a highly maneuverable homing missile can greatly improve the aircraft's survivability, making it possible to design relatively inexpensive defending missiles, without superior maneuverability characteristics, for such a mission.

Appendix

In this Appendix we present an example of the explicit kinematics for the case of zero-lag dynamics, taking into account the defender's disappearance after its interception with the attacking missile.

The state vector of the linearized problem for zero-lag dynamics is

$$\boldsymbol{x}(t) = [y_{MT}(t), \dot{y}_{MT}(t), y_{MD}(t), \dot{y}_{MD}(t)]^T \tag{A1}$$

and the equations of motion are

$$\dot{\boldsymbol{x}}(t) = \begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = (u_T(t) - u_M(t))1(t, t_{fMT}) \\ \dot{x}_3(t) = x_4(t) \\ \dot{x}_4(t) = (u_M(t)\cos(\lambda_{MT_0} - \lambda_{MD_0}) - u_D(t))1(t, t_{fMD}) \end{cases} \tag{A2}$$

where $1(t, t_{fMT})$ and $1(t, t_{fMD})$ are the indicator functions:

$$1(t, t_{fi}) = \begin{cases} 1, & t \le t_{fi} \\ 0, & t > t_{fi} \end{cases}, \qquad i = \{MT, MD\} \tag{A3}$$

which define the periods of the players' influence on their dynamics. After the interception of the attacking missile by the defending one ($t \ge t_{fMD}$), we assume that the defender ceases to exist. Therefore, the defending and attacking missiles' controls no longer influence their relative displacements. In the same way, for the sake of generality, we define the termination of the existence of the attacking missile and the evading target.

Consequently, the dynamic equation matrices from Eqs. (13) and (14) are defined as follows:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{B}(t) = \begin{bmatrix} 0 & 0 \\ 1(t, t_{fMT}) & 0 \\ 0 & 0 \\ 0 & -1(t, t_{fMD}) \end{bmatrix}$$

$$\mathbf{C}(t) = \begin{bmatrix} 0 \\ -1(t, t_{fMT}) \\ 0 \\ \cos(\lambda_{MT_0} - \lambda_{MD_0})1(t, t_{fMD}) \end{bmatrix} \tag{A4}$$

Integrating the first pair of equations from Eq. (A2) for $t \ge t_{fMT}$, we obtain

$$\begin{cases} x_2(t) = x_2(t_{fMT}) = \dot{y}_{MT}(t_{fMT}) = \text{const} \\ x_1(t) = x_1(t_{fMT}) + x_2(t_{fMT})(t - t_{fMT}), \quad t \ge t_{fMT} \\ = y_{MT}(t_{fMT}) + \dot{y}_{MT}(t_{fMT})(t - t_{fMT}) \end{cases} \tag{A5}$$

and, integrating the second pair of equations for $t \ge t_{fMD}$, we obtain

$$\begin{cases} x_4(t) = x_4(t_{fMD}) = \dot{y}_{MD}(t_{fMD}) = \text{const} \\ x_3(t) = x_3(t_{fMD}) + x_4(t_{fMD})(t - t_{fMD}), \quad t \ge t_{fMD} \\ \quad = y_{MD}(t_{fMD}) + \dot{y}_{MD}(t_{fMD})(t - t_{fMD}) \end{cases} \tag{A6}$$

Therefore, substituting Eq. (A6) in Eq. (47) we show that $Z_{MD}(t) \equiv_{t \ge t_{fMD}} Z_{MD}(t_{fMD})$. Similarly, $Z_{MT}(t) \equiv_{t \ge t_{fMT}} Z_{MT}(t_{fMT})$. Moreover, the ZEMs derivatives are

$$\begin{cases} \dot{Z}_{MT}(t) = -(t_{fMT} - t)1(t, t_{fMT})u_M + (t_{fMT} - t)1(t, t_{fMT})u_T \\ \quad = -t_{\text{go}MT}u_M + t_{\text{go}MT}u_T \\ \dot{Z}_{MD}(t) = (t_{fMD} - t)1(t, t_{fMD})u_M - (t_{fMD} - t)1(t, t_{fMD})u_D \\ \quad = t_{\text{go}MD}u_M - t_{\text{go}MD}u_D \end{cases} \tag{A7}$$

From these equations we obtain the nonnegative definitions of $t_{\text{go}MT}$ and $t_{\text{go}MD}$, as in Eq. (46). Furthermore, note that $\dot{Z}_{MT}(t)$ and $\dot{Z}_{MD}(t)$ are continuous for any $0 \le t \le t_{fMT}$.

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